Bosonization.

I.] Idea of the method.

Collective excitations.

Represent everything in terms of collective excitations

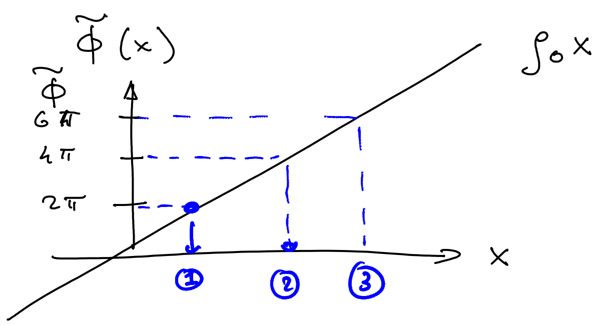
II.] Bosons :

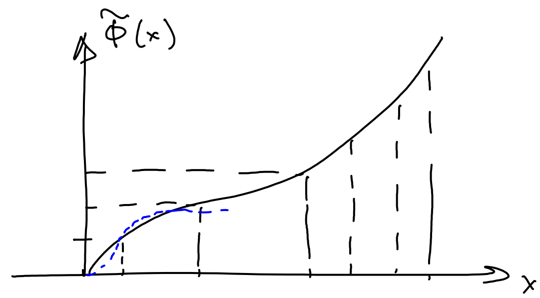
$$H = \frac{1}{2m}\int dx \; \nabla\psi^\dagger(x)\nabla\psi(x) \quad + \quad \int dx\,dx' \; V(x-x')\,\rho(x)\rho(x')$$

[bosons in continuum]

$V(x-x') = V_0\,\delta(x-x')$
$\hookrightarrow$ Lieb - Liniger model

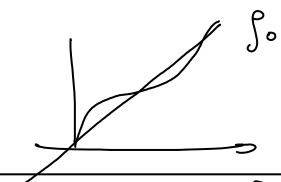
$$\rho(x) = \sum_i \delta(x - x_i)$$





$$\rho(x) = \sum_i \delta(x - x_i) \quad \longrightarrow \quad \sum_n |\nabla\tilde\phi(x)|\,\delta(\tilde\phi(x) - 2\pi n)$$

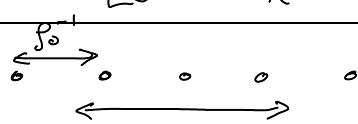
$$\rho(x) = \nabla\tilde\phi(x)\,\frac{1}{2\pi}\sum_p e^{ip\tilde\phi(x)}$$



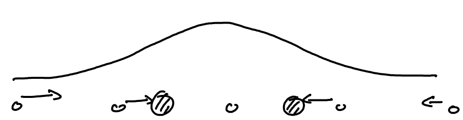
$$\tilde\phi(x) = 2\pi\rho_0 x - 2\phi(x)$$

$$\boxed{\rho(x) = \left[\rho_0 - \frac{1}{\pi}\nabla\phi(x)\right]\sum_p e^{ip\left[2\pi\rho_0 x - 2\phi(x)\right]}}$$

$\boxed{p=0}$



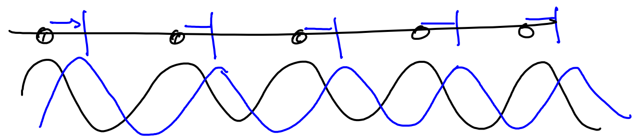
$$\rho_0 - \frac{1}{\pi}\nabla\phi(x)$$



$|p=0|$

$\rho_0 - \frac{1}{\pi} \nabla \phi(x)$

$\boxed{p=1}$ $\rho(x) \approx \rho_0 \, 2 \cos(2\pi\rho_0 x - \underbrace{2\phi(x)})$



ρ has no good continuum limit but $\phi(x)$ is a smooth field.

$$\boxed{\psi^\dagger(x) = [\rho(x)]^{1/2} e^{-i\Theta(x)}}$$

$$[\psi(x), \psi^\dagger(x')] = i\,\delta(x-x') \Rightarrow [\phi(x), \underbrace{\tfrac{1}{\pi}\nabla\Theta(x')}_{\Pi_\phi(x')}] = i\,\delta(x-x')$$

express everything : $\phi(x)$ $\Theta(x)$

$$H = \frac{1}{2m}\int dx \, \nabla\psi^\dagger(x)\nabla\psi(x) \approx \frac{\rho_0}{2m}\int dx \, (\nabla\Theta)^2 .$$

$$H_{int} = V_0 \int dx \, [\rho(x)]^2$$

$\hookrightarrow \rho_0 - \frac{1}{\pi}\nabla\phi$

$$V_0 \int dx \, \frac{1}{\pi^2} (\nabla\phi)^2 .$$

interacting boson hamiltonian.

$$\hookrightarrow \frac{\rho_0}{2m}\int dx \, \underbrace{(\nabla\Theta)^2}_{\Pi_\phi^2} + \frac{V_0}{\pi^2}\int dx \, (\nabla\phi)^2 .$$

$$= \int dx \left[\frac{\rho_0}{2m} \Pi_\phi^2(x) + \frac{V_0}{\pi^2} (\nabla\phi)^2 \right]$$

interaction.! $\int_0^L dx \, \nabla\phi(x)$

More General. Concept of Luttinger liquid.

All low energy properties are described by

$$H = \frac{1}{\ } \int dx \left[u K (\pi \Pi_\ )^2 + \frac{u}{\ } (\nabla\phi)^2 \right]$$

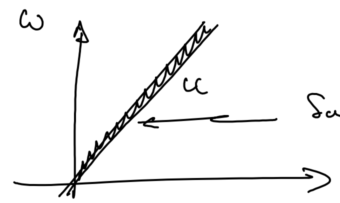
$$H = \frac{1}{2\pi}\int dx \left[uK \, (\pi \Pi_\phi)^2 + \frac{u}{K} (\nabla\phi)^2 \right]$$

u : velocity
K : dimensionless.
} Luttinger parameters

$\Rightarrow$ Compute $\underline{\underline{\text{all}}}$ correlation function.

$$S = \frac{1}{2\pi}\frac{1}{K}\int dx \left[\frac{1}{u} (\partial_\tau \phi)^2 + u (\nabla\phi)^2 \right]$$

$$\frac{1}{u}\omega^2 + u k^2 \qquad \boxed{\omega = uk}$$



Sound waves of density oscillations

$$C_v \simeq A \frac{1}{u} T$$

$$\langle \psi(x,\tau) \psi^\dagger(0,0) \rangle \equiv \rho_0 \langle e^{i\theta(x,\tau)} e^{-i\theta(0,0)} \rangle + \cdots \text{ higher harmonics.}$$

$$\simeq \rho_0 \; e^{-\frac{1}{2K} \text{Log}\left[\frac{\sqrt{x^2+(u\tau)^2}}{\alpha}\right]}$$

$$\simeq \left(\frac{1}{r}\right)^{1/2K} \qquad r = \sqrt{x^2+(u\tau)^2}.$$

$$\langle (\rho(x,\tau)-\rho_0)(\rho(0,0)-\rho_0) \rangle = \frac{K}{2\pi^2} \frac{(u\tau)^2 - x^2}{(x^2+(u\tau)^2)^2}$$

$$+ \rho_0^2 \cos(2\pi\rho_0 x) \left(\frac{1}{r}\right)^{2K}.$$

$$+ \cdots \cos(4\pi\rho_0 x) \left(\frac{1}{r}\right)^{8K}$$

$$+ \cdots$$

$\langle \nabla\phi \nabla\phi \rangle$

$\langle e^{i2\phi(x\tau)} e^{-i2\phi(00)} \rangle$

$\langle e^{i4\phi(x\tau)} e^{-i4\phi(00)} \rangle$

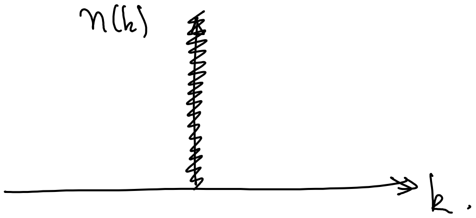
$$n_k = \int dx \; e^{ikx} \langle \psi(x) \psi^\dagger(0) \rangle \qquad \nu = \frac{1}{2K}.$$

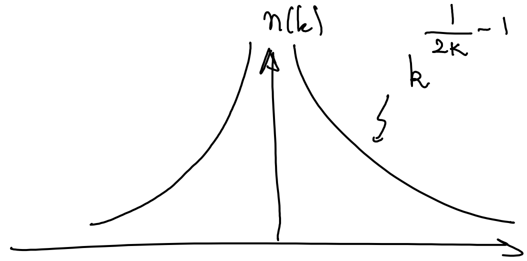
$$\left(\frac{1}{x}\right)^{\nu}$$

$$\sim \left(\frac{1}{k}\right)^{1-\nu} \sim k^{\frac{1}{2K}-1}$$

$n(k)$ $\quad \frac{1}{2K}-1$

$$\sim \left(\frac{1}{k}\right)^{1-\nu} \sim k^{\frac{1}{2K}-1}$$





$$\chi(q,\omega) \simeq \int dx\, d\tau\; e^{i(kx+\omega\tau)} \langle \underbrace{\rho(r)\rho(0)}_{(1/r)^{\nu}} \rangle$$

$$\sim r^{2-\nu} \approx \left(\mathrm{Max}[k,\omega]\right)^{\nu-2}$$

↳ Book / Lecture notes $\qquad \sim \mathrm{Max}(k,\omega)^{2K-2}$

$$\langle \rho(r)\rho(0)\rangle \sim \cos(2\pi\rho_0 x)\left(\frac{1}{r}\right)^{\nu}$$

$$\chi(Q \simeq 2\pi\rho_0 + \delta q) \qquad k = \delta q$$

$$\langle \psi\psi^\dagger\rangle \sim \left(\frac{1}{r}\right)^{\frac{1}{2K}} \qquad \langle \rho\rho\rangle = \cos(2\pi\rho_0 x)\left(\frac{1}{r}\right)^{2K}$$

III] Spins

$$H = J\sum_{\langle ij\rangle}\left(S_i^x S_j^x + S_i^y S_j^y\right) + J_z \sum_{\langle ij\rangle} S_i^z S_j^z$$

$$H = \frac{J}{2}\sum_{\langle ij\rangle}\left(b_i^\dagger b_j + b_j^\dagger b_i\right) + J_z\sum_{\langle ij\rangle}\left(n_i - \tfrac{1}{2}\right)\left(n_j - \tfrac{1}{2}\right)$$

$$+ U\sum_i n_i(n_i - 1)$$

$U \to \infty$

$$S_i^+ = b_i^\dagger \qquad S_i^- = b_i \qquad S_z = n_i - \tfrac{1}{2}$$

$$\langle \delta S_z \delta S_z\rangle = \frac{1}{x^2} + \underbrace{\cos(\pi x)}_{(-1)^x}\left(\frac{1}{r}\right)^{2K}.$$

$$S_i^+ = (-1)^i\left[e^{i\theta} + e^{i\theta}\underbrace{e^{i(2\pi\rho_0 x - 2\phi)}}_{(-1)^i} + \cdots\right]$$

$$S_i = (-1) \Big[e \quad + \quad e \; \underbrace{e}_{(-1)^i} \quad + \cdots \Big]$$

$$\begin{cases} \langle S_r^+ S_0^- \rangle = \left(\frac{1}{r}\right)^{2K + \frac{1}{2K}} + (-1)^x \left(\frac{1}{r}\right)^{\frac{1}{2K}} \\ \langle S_z S_z \rangle = \frac{1}{x^2} + (-1)^i \left(\frac{1}{r}\right)^{2K} \end{cases}$$

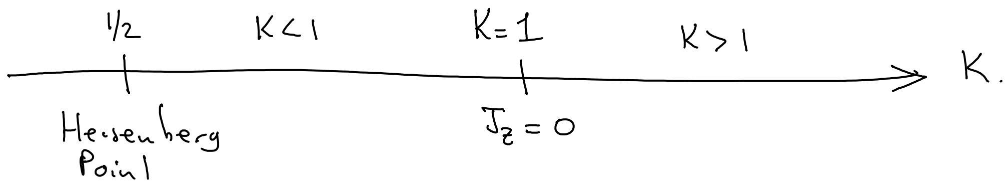
Heisenberg $\rightarrow$ $J_{xy} = J_z$ $\qquad K = 1/2$

[Log corrections $\left(\frac{1}{r}\right) \operatorname{Log}^{1/2} r$

PHYSICAL REVIEW B VOLUME 39, NUMBER 7 1 MARCH 1989

Correlation functions of one-dimensional quantum systems

T. Giamarchi and H. J. Schulz]



IV.] Fermions (Spinless):

$$S_i^+ = f_i^+ \, e^{i\pi \sum_{j<i} \underbrace{f_j^+ f_j}_{S_z}}$$

$$f_i^+ = \underbrace{S_i^+}_{b_i^+} \, e^{i\pi \sum_{j<i} b_j^+ b_j}$$

$$\longrightarrow \; -\frac{1}{\pi} \nabla\phi + e^{i(2\pi\rho_0 x - \cdots)}$$

$$\int_{-\infty}^{x} dy \left[-\frac{1}{\pi} \nabla\phi + \cancel{\underbrace{e^{i 2\pi\rho_0 y} \cdots}} \right.$$

$$f_i^+ = b_i^+ \, e^{i\phi(x)}$$

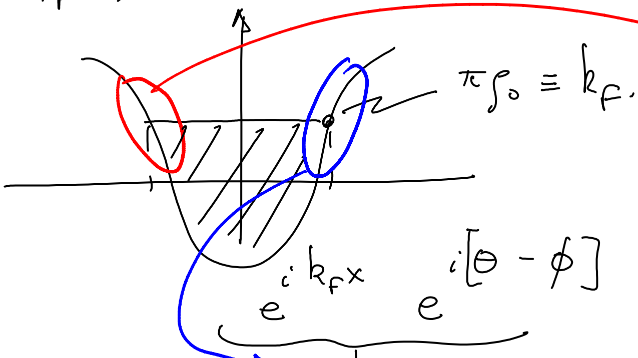
$$\psi_B(x) = e^{i\theta} \sum_p e^{i 2p (\pi\rho_0 x - \phi(x))}$$

$$\psi_F(x) = e^{i\theta} \sum_p e^{i(2p+1)(\pi\rho_0 x - \phi(x))}$$

$\psi(x) \sim e^{i\theta}$ + higher harmonics

$$\psi_B(x) \approx e^{i\theta} + \text{higher harmonics}$$

$$\psi_F(x) = e^{i\theta + \pi\rho_0 x - \phi} + e^{i\theta - \pi\rho_0 x + \phi}$$



$$\underbrace{e^{i k_F x}\, e^{i[\theta - \phi]}}_{\psi_R^+(x)} + \underbrace{e^{-i k_F x}\, e^{i[\theta + \phi]}}_{\psi_L^+(x)}$$

$$\rho(x) = \rho_0 - \frac{1}{\pi}\nabla\phi + e^{i2(\pi\rho_0 x - 2\phi)} + \cdots\cdots$$

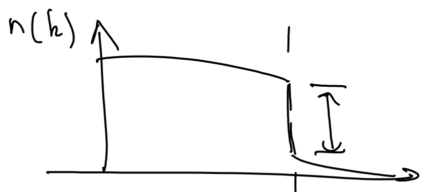
$$\langle \rho(x)\rho(0)\rangle = \frac{1}{x^2} + \mathrm{Cos}(2k_F x)\left(\frac{1}{r}\right)^{2K}$$

$K = 1$

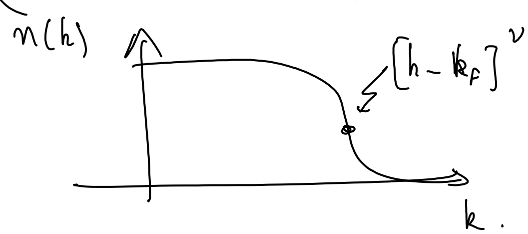
$$\langle \psi_F(x)\psi_F^+(0)\rangle = e^{ik_F x}\left(\frac{1}{r}\right)^{\frac{1}{2}\left[K + \frac{1}{K}\right]}\left(\cdots\cdots\right)$$

$e^{i(\theta-\phi)}$

$$n(k) \simeq \int dx\, e^{i(k-k_F)x}\left(\frac{1}{r}\right)^{\frac{1}{2}\left[K+\frac{1}{K}\right]} \simeq [k-k_F]^{\frac{1}{2}\left[K+\frac{1}{K}\right]-1}$$



Fermi Liquid



VI] Additional stuff on correlation functions

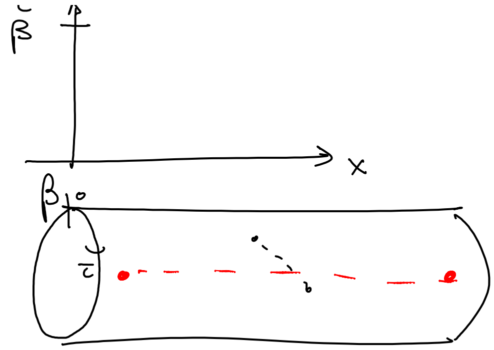
Finite temp

$$\omega \longrightarrow \omega_n = \frac{2\pi}{\beta} n.$$

$$\langle \rho(r)\rho(0)\rangle \sim \left(\frac{1}{r}\right)^{2K} \Longrightarrow e^{-r/\boxed{\xi}}$$

τ β $\uparrow$

Conformal invariance

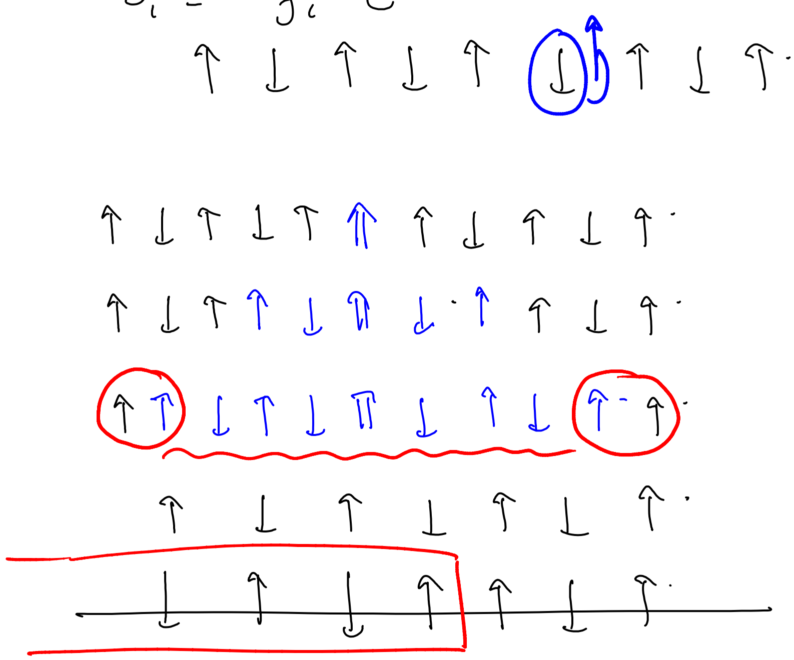


Conformal invariance

$\xi = \beta f(K)$

Excitations in 1D

$S_i^+ = f_i^+ e^{i\pi \Sigma \cdots}$



$\Delta S_z = 1$

Magnon.

$\omega(q) = [1 - \cos^2(q)]^{1/2}$

Spinons

$\Delta S_z = 1/2$

VI] Calculation of Luttinger parameter VI]

$C_v \sim \frac{1}{u} T$

$K = \frac{\partial N}{\partial \mu} = \frac{u}{K}.$

$\psi(L) = e^{i\phi} \psi(0) \Rightarrow \frac{\partial^2 E_0}{\partial \phi^2} = u_0 K$

Luttinger

$\langle \psi(x) \psi(0) \rangle \sim \left(\frac{1}{x}\right)^{\nu}$

$\frac{E(L) - E(\infty)}{L} = \cdot \; c \; u \frac{1}{L^2}$, $c = 1$

example : Spin chain $(m=0)$

Bethe ansatz

$$
\begin{aligned}
J_z/J_{xy} &= -\cos \pi\beta^2 \\
1/K &= 2\beta^2 \\
u &= \frac{1}{1-\beta^2}\sin(\pi(1-\beta^2))\frac{J_{xy}}{2}
\end{aligned}
$$

More details (or program) → Book 1D